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ON ARITHMETIC VARIETIES II

BY
DAVID KAZHDAN[†]

ABSTRACT
An arithmetic variety is the quotient space of a symmetric space with complex structure by an arithmetic subgroup of the associated algebraic Lie group. It is shown that the variety obtained from an arithmetic variety by a base change corresponding to any automorphism of C is again an arithmetic variety.

Introduction

Let G be an algebraic simple $\mathbf{Q}$-group, $G_{\mathbf{R}}$ the set of real points of G and $K \subset G_{\mathbf{R}}$ a maximal compact subgroup.

We assume that there is a $G_{\mathbf{R}}$-invariant complex structure on the symmetric space $D = K \setminus G_{\mathbf{R}}$ and we will always consider D as a complex manifold. Let $\Gamma \subset G_{\mathbf{R}}$ be an arithmetic subgroup without nontrivial elements of finite order and let $X_\Gamma \stackrel{\text{def}}{=} D/\Gamma$. X_Γ has the natural structure of a smooth complex manifold. Such complex manifolds $X = X_\Gamma$ will be called arithmetic varieties. It is known ([1], [11]) that there exists an imbedding $X \to \mathbf{P}^N(\mathbf{C})$ such that the closure $\bar{X}$ of X is a normal variety, and $Y = \bar{X} - X$ is a subvariety in $\mathbf{P}^N$. Moreover, if $\dim D > 1$ then a multicanonical bundle $\Omega^{\otimes k}$ on X defines such an imbedding and in this case codim $Y > 1$. We will assume that $\dim D > 1$ and that an imbedding $X \to \mathbf{P}^N$ is a multicanonical one. We will call $\bar{X}$ "the canonical completion of X".

By Chow's Theorem $\bar{X}$ is algebraic and we will denote by $\bar{X}_a$ the corresponding algebraic $\mathbf{C}$-varieties.

For any $\sigma \in \operatorname{Aut} \mathbf{C}$ we denote by $\bar{X}_a^\sigma$ the algebraic variety obtained from $\bar{X}_a$ by the base change and by $\bar{X}^\sigma$ the complex variety of $\mathbf{C}$ -points of $\bar{X}_a^\sigma$. We denote by $X^\sigma \subset \bar{X}^\sigma$ the open subvariety which corresponds to $X_a^\sigma \stackrel{\text{def}}{=} \bar{X}_a^\sigma - \bar{Y}_a^\sigma$.

MAIN THEOREM. *For any $\sigma \in \operatorname{Aut} \mathbf{C}$ the variety X^σ is arithmetic.*

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It is easy to see (lemma 0 in [5]) that it suffices to prove the Main Theorem for torsion free subgroups.

Let M be a reductive $\mathbf{Q}$-group and $\varphi : H \rightarrow G$ be an algebraic morphism. We say that φ is *admissible* if

(a) $K_H \overset{\text{def}}{=} \varphi_{\mathbf{R}}^{-1}(\varphi_{\mathbf{R}}(H_{\mathbf{R}}) \cap K)$ is a maximal compact subgroup of $H_{\mathbf{R}}$.

(b) $\bar{\varphi} : D_H \rightarrow D$ is an imbedding where $D_H \overset{\text{def}}{=} K_H \setminus H_{\mathbf{R}}$.

(c) The image $\bar{\varphi}(D_H)$ is a complex submanifold of D.

We say that G is a classical group if there exists an admissible morphism $\varphi : G \rightarrow \mathrm{Sp}(2N,\quad)$ for some N. It is proved in [2] and [3] that

(a) The Main Theorem is true for classical groups.

(b) If G is a simple, nonclassical $\mathbf{Q}$-group, then $G = R_{k/\mathbf{Q}}(\tilde{G})$ where k is a totally real number field, $\tilde{G}$ is an absolutely simple k-group of type D_e, $e \geqq 4$ and $G_{\mathbf{R}}$ has factors of types $D_e^{\mathbf{R}}$ and D_e^H or $\tilde{G}$ is of type E_6 or E_7 and G is obtained from $\tilde{G}$ by restriction of the scalar field from k.

In the next paragraph we outline our approach to the proof of the Main Theorem and give a very sketchy proof in the classical case.

REMARK. This theorem was proved in [5] for anisotropic groups and our proof will go along the same lines. As [5] is not an easily readable paper we will try to refer only to the first two paragraphs of [5]. We start by recalling some general facts.

NOTATION. (1) For any algebraic $\mathbf{C}$-variety X we denote by $X_{\mathbf{C}}$ the set $\mathbf{C}$-points of X considered as an analytic variety.

(2) For any analytic variety X we denote by $G(X)$ the group of analytic transformations of X.

(3) For any manifold X and a point $x \in X$ we denote by $T_X(x)$ the tangent space to X at x.

(4) For an arithmetic variety X we denote by $\bar{X}$ the canonical completion of X, by Y the complement $Y = \bar{X} - X$ and by $j : X \hookrightarrow \bar{X}$ the natural imbedding.

(5) For any group G we denote by e the identity element.

(6) For any group G acting on a set S and a subset $T \subset S$ we denote by $\mathrm{St}_G(T) \subset G$ the stabilizer of T in G.

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§1. We start by recalling some concepts and results from [5]. For any $\sigma \in \mathrm{Aut}\,\mathbf{C}$ we consider X^σ the complex variety corresponding to an algebraic

C-variety X_a^σ. We denote by Ω the intersection of all subgroups of finite index in $\pi_1(X^\sigma)$, by the $D^\sigma \xrightarrow{p_\sigma} X^\sigma$ covering of X^σ corresponding to Ω and by $\Gamma^\sigma \overset{\text{def}}{=} \pi_1(X^\sigma)/\Omega$ the Galois group of this covering. We denote by $G(D^\sigma)$ the group of analytic transformations of D^σ and identify Γ^σ with a subgroup of $G(D^\sigma)$.

For any $g \in G_{\mathbf{Q}}$ we consider the complex variety $R_g \overset{\text{def}}{=} D/\Gamma_g$, for $\Gamma_g = \Gamma \cap g^{-1}\Gamma g$ and two projections

$$R_g \overset{q_1}{\underset{q_2}{\rightrightarrows}} X$$

which correspond to two evident imbeddings

$$\Gamma_g \overset{i_1}{\underset{i_2}{\rightrightarrows}} \Gamma.$$

It is clear that q_1, q_2 are finite unramified coverings of X. Therefore there exists a unique algebraic **C**-variety $\hat{R}_g$ and two unramified coverings

$$\hat{R}_g \overset{\hat{q}_1}{\underset{\hat{q}_2}{\rightrightarrows}} \hat{X}$$

such that

$$\left(R_g \overset{q_1}{\underset{q_2}{\rightrightarrows}} X\right) = (\hat{R}_g)_{\mathbf{C}} \overset{(\hat{q}_1)_{\mathbf{C}}}{\underset{(\hat{q}_2)_{\mathbf{C}}}{\rightrightarrows}} (\hat{X})_{\mathbf{C}}.$$

For any $\sigma \in \operatorname{Aut} \mathbf{C}$ we denote by

$$R_g^\sigma \overset{q_1^\sigma}{\underset{q_2^\sigma}{\rightrightarrows}} X^\sigma$$

the diagram of complex varieties corresponding to

$$\hat{R}_g^\sigma \overset{\hat{q}_1^\sigma}{\underset{\hat{q}_2^\sigma}{\rightrightarrows}} \hat{X}^\sigma.$$

Of course q_1^σ, q_2^σ are finite unramified coverings of X^σ.

LEMMA. *Construction* 1. (a) *R_g^σ defines some double Γ^σ-coset M_g in $G(D^\sigma)$.*
(b) *$\bigcup_{g \in G_{\mathbf{Q}}} M_g$ is a subgroup in $G(X^\sigma)$ which we will denote by G^σ.*

The construction and the proof are contained in §1 of [5].

Let $H \xrightarrow{\varphi} G$ be an admissible morphism, $\Gamma \subset G_{\mathbf{Q}}$ be an arithmetic group, $\Gamma_H \overset{\text{def}}{=} \varphi_{\mathbf{Q}}^{-1}(\varphi_{\mathbf{Q}}(H_{\mathbf{Q}}) \cap \Gamma)$. It is clear that Γ_H is an arithmetic subgroup in $H_{\mathbf{R}}$ and $\tilde{\varphi}: D_H \to D$ identifies $X_H \overset{\text{def}}{=} D_H/\Gamma_H$ with an analytic subvariety in $X = D/\Gamma$. It is well known that there exists an algebraic subvariety $X_{H,a}$ in X_a such that $X_H = (X_{H,a})_{\mathbf{C}}$. Let $X_H^\sigma \overset{\text{def}}{=} (X_{H,a}^\sigma)_{\mathbf{C}} \subset X^\sigma$ be the corresponding subvariety in X^σ and $'D_H^\sigma$ be a connected component of $(p^\sigma)^{-1}(X_H^\sigma)$ in D^σ.

LEMMA 2. (a) *For any $h \in H_{\mathbf{Q}}$ the intersection $M_h \overset{\text{def}}{=} M_{\varphi(h)} \cap \mathrm{St}_{G(X^\sigma)}('D_H^\sigma)$ is not empty.*

(b) *The union $'H^\sigma \overset{\text{def}}{=} \bigcup_{h \in H_{\mathbf{Q}}} {}'M_h$ is a subgroup in G^σ.*

(c) *There exists a normal subgroup $\Lambda \subset H^\sigma$ such that $\Lambda \subset \Gamma_H^\sigma$, $'H^\sigma \simeq H^\sigma/\Lambda$ and $'D_H^\sigma \simeq D_H^\sigma/\Lambda$.*

PROOF. Clear. □

COROLLARY. *If the Main Theorem is true for H then $'D_H^\sigma$ is a Hermitian symmetric space and $'H^\sigma$ is dense in $G('D_H^\sigma)$.*

PROOF. Also clear. □

PROPOSITION 1. *Assume that:*

(a) *The Bergman metric (see [6]) ρ_{D^σ} is not degenerate.*

(b) *The closure $\bar{G}^\sigma$ of G^σ in $G(D^\sigma)$ acts transitively on D^σ.*

(c) *There exists a G^σ-invariant volume form μ_{D^σ} on D^σ such that $\int_{X^\sigma} \mu_{D^\sigma} < \infty$.*
Then X^σ is an arithmetic variety.

PROOF. By the assumption D^σ is a homogeneous complex variety with a nondegenerate Bergman metric. By [11] D^σ is a Hermitian symmetric space $D^\sigma = K^\sigma \setminus G_{\mathbf{R}}$. It is clear that $d\mu_{D^\sigma}$ is $G_{\mathbf{R}}^\sigma$ invariant volume ν on D^σ. Therefore

$$\nu(K^\sigma \setminus G_{\mathbf{R}}^\sigma / \Gamma^\sigma) = \int_{X^\sigma} d\mu_{D^\sigma} < \infty.$$

It follows now from Margulis's theorem ([8]) that $\Gamma^\sigma \subset G_{\mathbf{R}}^\sigma$ is an arithmetic subgroup. □

For $G = \mathrm{Sp}(2n, \quad)$ the Main Theorem was known long ago (in this case X may be interpreted as a moduli space M for polarized abelian varieties with additional rigidity structure and $M^\sigma = M$ for all $\sigma \in \mathrm{Aut}\,\mathbf{C}$ if Γ is an appropriately chosen arithmetic subgroup).

If G is a classical group, then an admissible morphism $\varphi : G \to \mathrm{Sp}(2N, \quad)$ induces the imbedding $\varphi : X \to M$. Therefore we have $X^\sigma \to M^\sigma \simeq M$ and $'D_G^\sigma \subset \mathcal{H}$ where $\mathcal{H}$ is the Siegel upper "halfplane". In this case, instead of the Bergman metric ρ_{D^σ} on $'D_D^\sigma$ we can take the restriction of the Bergman metric $\rho_{\mathcal{H}}$ on $'D_\rho$. Although we cannot apply Proposition 1, we could finish the proof by making use of arguments from the second part of [5]. In any case, the Main Theorem is known [3] for classical groups.

Unfortunately for nonclassical groups, no modular interpretation for X is known and we will examine the conditions of Proposition 1 directly.

REMARK. In the case when X is compact we can easily prove the Main Theorem using the Calabi–Einstein metric on X([16]).

§2. We will assume from now on that a simple isotropic **Q**-group G is of type $R_{k/\mathbf{Q}}(\tilde{G})$ where k is a totally real number field and $\tilde{G}$ is an absolutely simple k-group of type D_l, $l > 3$, such that $G_{\mathbf{R}}$ has $D_l^{\mathbf{R}}$ and $D_l^{\mathbf{H}}$ factors or $\tilde{G}$ is an absolutely simple k-group of type E_6 or E_7. We will call such G "our groups". As before, let K be a maximal compact subgroup in $G_{\mathbf{R}}$, $D = K \setminus G$ be the corresponding Hermitian symmetric space, $\Gamma \subset G_{\mathbf{Q}}$ be an arithmetic subgroup without elements of finite order, $X = D/\Gamma$ be the corresponding arithmetic variety, $\bar{X}$ be the canonical compactification of X and $Y = \bar{X} - X$.

PROPOSITION 1. *Let G be one of our groups. Then* (a) codim $Y > 3$, (b) *there exists a semisimple* **Q**-*group H and an admissible morphism $\varphi : H \to G$ such that* $\dim Y < \dim D_H < \dim X$ *and there exists a torus $T \subset H$ such that* $(T_{\mathbf{R}})$ *is a compact Cartan subgroup of $G_{\mathbf{R}}$.*

We start the proof with the following

LEMMA 1. *Let G be one of our groups $G = R_{k/\mathbf{Q}}(\tilde{G})$. Then*

(α) *If G is of type D_l and a be the number of imbeddings $k \xrightarrow{i} \mathbf{R}$ such that the corresponding real group $\tilde{G}_{\mathbf{R}}$ is of type $D_l^{\mathbf{R}}(\approx$ SO $(2l-2,2))$ and b be the number of imbeddings $k \xrightarrow{i} \mathbf{R}$ such that $\tilde{G}_{\mathbf{R}}$ is of type $D_l^{\mathbf{H}}(\simeq$ SO*(2)), then*

$$\dim_{\mathbf{C}} X = a(2l-2) + b\frac{l(l-1)}{2} \quad \textit{and} \quad \dim_{\mathbf{C}} Y \leqq a + b\frac{(l-2)(l-3)}{2}.$$

(β) *If $\tilde{G}$ is of type E_6, then*

$$\dim X = 16 \cdot [k : \mathbf{Q}] \quad \textit{and} \quad \dim Y \leqq 5[k : \mathbf{Q}].$$

(γ) *If $\tilde{G}$ is of type E_7, then*

$$\dim X = 27[k : \mathbf{Q}] \quad \textit{and} \quad \dim Y \leqq 10[k : \mathbf{Q}].$$

This lemma follows immediately from theorem 4.13 in [7]. □

Part (a) from Proposition 1 follows immediately from this lemma.

If $G = R_{k/\mathbf{Q}}(\tilde{G})$ is a simple **Q**-group and x is a distinguished vertex of the Dynkin diagram of $\tilde{G}$ we denote by $\tilde{M}_x \subset \tilde{G}$ the semisimple part of the Levy component L_x of the parabolic subgroup P in $\tilde{G}$ which corresponds to x. (See [15].) Let $\tilde{V}_x = \text{Centralizer}_{\tilde{G}}(\tilde{M}_x)$, $H_x = R_{k/\mathbf{Q}}(\tilde{M}_x \cdot \tilde{V}_x)$.

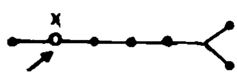
Lemma 2. *If $D = K \backslash G_{\mathbf{R}}$ is a Hermitian symmetric space and* $\dim_k \tilde{V}_x > 1$, *then the natural imbedding $H_x \to G$ is an admissible morphism.*

Proof of Lemma 2. It is clear that $\tilde{V}_x$ is a reductive k-subgroup in $\tilde{G}$. Let $\bar{k} = \bar{\mathbf{Q}}$ be the algebraic closure of k. Since $\mathrm{rk}_{\bar{k}}(\tilde{M}_x) = \mathrm{rk}_{\bar{k}}(\tilde{G}) - 1$, we know that $\mathrm{rk}_{\bar{k}}(\tilde{V}_x) \leqq 1$. Therefore $\tilde{V}_x$ is a k-form of A_1.

Consider $V = R_{k/\mathbf{Q}}\tilde{V}_x$ and $M = R_{k/\mathbf{Q}}\tilde{M}_x$. Let S_1 be a maximal compact torus in $V_{\mathbf{R}}$ and S_2 be the one in $M_{\mathbf{R}}$. It is clear that $S \overset{\text{def}}{=} S_1 S_2$ is a maximal compact torus in $G_{\mathbf{R}}$. Since the k-variety of maximal tori in $\tilde{G}$ is a k-rational variety (see exp. XIV, th. 6.2 in SGA3), it is clear now that there exists a $\mathbf{Q}$-torus $T \subset H_x$ such that $T_{\mathbf{R}} \subset G_{\mathbf{R}}$ is a maximal compact torus in $G_{\mathbf{R}}$. We take $K \subset G_{\mathbf{R}}$ to be the maximal compact subgroup containing $T_{\mathbf{R}}$. Then tangent subspace $T_{D_{H_x}}(e) \subset D_D(e)$ to $D_{H_x} \subset D$ is invariant under multiplication by $\sqrt{-1}$. Therefore the same is true for all points $d \in D_{H_x}$ and D_{H_x} is a complex submanifold of D. Lemma 2 is proved. ❑

To finish the proof of Proposition 1 we will point out for any of our groups G a vertex x satisfying conditions of Lemma 2 and such that $\dim D_{H_x} > \dim Y$. We will be using the classification of simple k-groups from [15].

If $\tilde{G}$ is an isotropic k-group of type D_l and

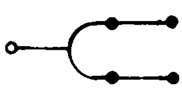


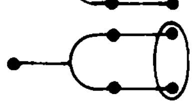
is the Dynkin diagram of G, then the vertex x second from the left is a distinguished one, and (as follows from the shape of the extended Dynkin diagram for D_l) x satisfies conditions of Lemma 2. It is clear that

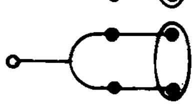
$$\dim D_{H_x} = 2a + b\left(\frac{(l-3)(l-2)}{2} + 1\right) > \dim Y.$$

(For $l = 4$ we take x to be the center of the Dynkin diagram.)

If $\tilde{G}$ is of type E_6, then it follows from the table on page 59 in [15] that the index of $\tilde{G}$ is either

${}^2E^{35}_{6,1}$  or

${}^2E^{29}_{6,1}$  or

${}^2E^{16'}_{6,2}$ 

LEMMA 3. *If $\tilde{G}$ is any k-group with index ${}^2E_{6,1}^{29}$ then the symmetric space of $\tilde{G}_{\mathbf{R}}$ is not Hermitian.*

PROOF. Let $H \subset G$ be the anisotropic kernel of $\tilde{G}$ ([15]). It acts on a maximal unipotent k-subgroup U of G. Since $\dim[U, U] = 8$, the action of H on $[U, U]$ identifies H with an orthogonal subgroup of $\mathrm{End}_k [U, U]$. Now it follows from the classical Hasse Principle that there exists a completion k_V of k such that $H \otimes_k \mathrm{Spec}\, k_V$ is anisotropic, and therefore $k_V \simeq \mathbf{R}$. The corresponding real group $L = \tilde{G} \otimes_k \mathrm{Spec}\, k_V$ has the index ${}^1E_{6,2}^{28}$ and the corresponding symmetric space is not Hermitian. As L is a factor of $\tilde{G}_{\mathbf{R}}$ the symmetric space of $\tilde{G}_{\mathbf{R}}$ also does not have an invariant complex structure. □

So we may restrict attention to groups with indices ${}^2E_{6,1}^{35}$ and ${}^2E_{6,2}^{16'}$. Let x be the left distinguished vertex. It is clear that x satisfies the conditions of Lemma 2 and $\dim D_{H_x} = 6 \cdot [k : \mathbf{Q}] > \dim Y$.

Consider now the case when $\tilde{G}$ is of type E_7. Then the index of $\tilde{G}$ is either $E_{7,2}^{31}$ or $E_{7,3}^{28}$. Let x be the right distinguished vertex. Then $D_{H_x} \subset D$ is a complex submanifold and $\dim D_{H_x} > \dim Y$. Proposition 1 is proved.

In the future, we will always assume that the Main Theorem is known for H.

§3. We will study in this paragraph different G^σ-invariant analytic objects on D^σ.

But in the beginning we will restate some results from [5].

For any $d \in D$ we denote by $G_d \subset G_{\mathbf{Q}}$ the stabilizer of d in $G_{\mathbf{Q}}$ and by χ_d the natural representation $\chi_d : G_d \to \mathrm{Aut}\, T_D(d)$.

Let $d^\sigma \in D^\sigma$ be a point such that $(p(d))^\sigma = (p^\sigma)(d^\sigma)$. Define $G_{d^\sigma} \subset G^\sigma$ to be the stabilizer of d^σ, let $\chi_{d^\sigma} : G_{d^\sigma} \to \mathrm{Aut}\, T_{D^\sigma}(d^\sigma)$ be defined in the same way as χ_d and denote by $\alpha_d : T_D(d) \to T_{D^\sigma}(d^\sigma)$ the composition

$$T_D(d) \xrightarrow{p_*} T_X(p(d)) \xrightarrow{\sigma} T_{X^\sigma}(p(d)^\sigma) \xrightarrow{p_*^{\sigma^{-1}}} T_{D^\sigma}(d^\sigma).$$

LEMMA 1. *There exists an isomorphism $\varphi_d : G_d \to G_{d^\sigma}$ such that $\alpha_d \circ \chi_d(\gamma) = \chi_{d^\sigma}(\varphi_d(\gamma)) \circ \alpha_d$ for all $\gamma \in G_d$.*

This lemma is also proven in §1 of [5].

DEFINITION. We say that $d \in D$ is a CM-point if there exists a maximal $\mathbf{Q}$-torus H in G such that $H_{\mathbf{Q}} \subset G_d$.

LEMMA 2. *If d is a CM-point in D, then the closure $\bar{H}^\sigma$ (in the usual topology) of $\chi_{d^\sigma}(G_{d^\sigma})$ in $\mathrm{Aut}\, T_{D^\sigma}(d^\sigma)$ contains the multiplication by $\sqrt{-1}$.*

PROOF. For any algebraic $\mathbf{Q}$-torus H we denote by $\hat{H}$ the group of algebraic characters of H over $\mathbf{C}$. $\hat{H}$ is a free abelian group with a natural action of $\mathrm{Aut}\,\mathbf{C}$. We denote it by $\sigma : \hat{h} \to \hat{h}^\sigma$ for $\sigma \in \mathrm{Aut}\mathbf{C}$, $\hat{h} \in \hat{H}$.

By definition we have $(\hat{h}(h))^\sigma = (\hat{h}^\sigma)(h)$ for any $\sigma \in \mathrm{Aut}\,\mathbf{C}$, $\hat{h} \in \hat{H}$ and $h \in H_{\mathbf{Q}}$. It is easy to observe (see, for example, lemma 18 in [5]) that for any $\sigma \in \mathrm{Aut}\,\mathbf{C}$ there exists an automorphism $\sigma : h \to h$ of $H_{\mathbf{C}}$ such that $\hat{h}^\sigma(h) = \hat{h}(h^\sigma)$ for all $\hat{h} \in \hat{H}$, $h \in H_{\mathbf{C}}$. Assume now that $H_{\mathbf{R}}$ is a compact group. Then it is a unique maximal compact subgroup of $H_{\mathbf{C}}$ and therefore $\tilde{\sigma}$ maps $H_{\mathbf{R}}$ onto itself for all $\sigma \in \mathrm{Aut}\,\mathbf{C}$. Let now $d \in D$ be a CM-point. We consider the representation $\chi_d = \chi_{d^\sigma} \circ \chi_d : H_{\mathbf{Q}} \to \mathrm{Aut}\; T_{D^\sigma}(d^\sigma)$. As follows from Lemma 1 there exists a $\mathbf{C}$-linear operator $c : T_D(d) \tilde{\to} T_{D^\sigma}(d^\sigma)$ such that $\chi_d^\sigma(h) = c \circ (\chi_d(h))^\sigma \circ c^{-1}$ for $h \in H_{\mathbf{Q}}$. We will consider $\chi_d : H_{\mathbf{Q}} \to \mathrm{Aut}\; T_D(d)$ as a restriction of an algebraic homomorphism $\chi_d : H_{\mathbf{C}} \to \mathrm{Aut}\; T_D(d)$ and rewrite $\chi_d^\sigma(h) = c \circ \chi_d(h^\sigma) \circ c^{-1}$.

Let $H_{\mathbf{R}}^0$ be the connected component of identity in $H_{\mathbf{R}}$ and $H_{\mathbf{Q}}^0 = H_{\mathbf{Q}} \cap H_{\mathbf{R}}^0$. $H_{\mathbf{Q}}^0$ is dense in $H_{\mathbf{R}}^0$. Therefore the closure $\bar{H}^\sigma$ of $\chi_{d^\sigma}(G_{d^\sigma})$ contains $c \circ \chi_d(H_{\mathbf{R}}^0) \circ c^{-1}$. Since $H_{\mathbf{R}}$ is compact $H_{\mathbf{R}}^\sigma = H_{\mathbf{R}}$ and $\bar{H}^\sigma \supset c \circ \chi_d(H_{\mathbf{R}}) \circ c^{-1}$. Since D admits an invariant complex structure there exists $h_0 \in H_{\mathbf{R}}$ such that $\chi_d(h_0) = \sqrt{-1}\,\mathrm{Id}$. The lemma is proved.

LEMMA 3. *Let $\tilde{Z} \subset D^\sigma$ be a nonempty complex analytic G^σ-invariant subvariety such that $Z \overset{\mathrm{def}}{=} p^\sigma(Z) \subset X^\sigma$ is algebraic. Then $Z = D^\sigma$.*

PROOF. The arguments which are used in the proof of lemma 3 in [5] are applicable here. □

COROLLARY 1. *Let $\tilde{Z} \subset D^\sigma$ be a complex irreducible G^σ-invariant analytic subvariety such that $\dim \tilde{Z} > \dim Y$, where as always $Y = \bar{X} - X$. Then $\tilde{Z} = D^\sigma$ or $\tilde{Z} = \varnothing$.*

PROOF. $Z = p^\sigma(\tilde{Z}) \subset X^\sigma$ is an irreducible analytic variety. Therefore $Z \cup Y^\sigma \subset \bar{X}^\sigma$ is $*$-analytic subvariety (cf. [10]) of $\bar{X}^\sigma$. By theorem 4.5 in [10] it is analytic and by Chow's Lemma algebraic subvariety in $\bar{X}^\sigma$. □

Consider now the real semisimple group $G_{\mathbf{R}}$, and write it as a product of simple groups $G_{\mathbf{R}} = \prod_{i=1}^s G_i$. Then $D = \prod_{i=1}^s D_i$ where D_i is the symmetric space for G_i and the tangent bundle T_D decomposes into the direct sum $T_D = \bigoplus_{i=1}^s T_{i,D}$. This decomposition induces the decomposition $T_X = \bigoplus_{i=1}^s T_{i,X}$. To prove algebraicity of $T_{i,X}$ we will use the following

LEMMA 4. *Let $\bar{M}$ be a normal projective algebraic variety, $M \subset \bar{M}$ be an open subset such that $N \overset{\mathrm{def}}{=} \bar{M} - M$ is analytic, $\mathrm{codim}_{\bar{M}} N > 1$ and all points of M are smooth.*

Let $W \subset T_M$ be an analytic subbundle. Then W is algebraic.

PROOF. Let $\bar{M} \to \mathbf{P}^N$ be a projective imbedding and $k = \{$Dimension of a fibre of $W\}$. Then W defines an analytic map

$$\varphi : M \to \mathrm{Gr}_{k,N} \stackrel{\text{def}}{=} \{\text{The variety of } k\text{-planes in } \mathbf{P}^N\}.$$

By [14] φ is extendable to a meromorphic map $\bar{\varphi} : \bar{M} \to \mathrm{Gr}_{k,N}$. By Chow's Lemma $\bar{\varphi}$ is algebraic. □

So $T_{i,X}$ is an algebraic subsheaf of X and we can define decompositions $T_{X^\sigma} = \bigoplus_{i=1}^s T_{i,X^\sigma}$ and $T_{D^\sigma} = \bigoplus_{i=1}^s T_{i,D^\sigma}$.

DEFINITION. Let N be a complex variety and $\sim$ be an equivalence relation on N. We say that $\sim$ is analytic if $\tilde{\Gamma} \stackrel{\text{def}}{=} \{(n_1, n_2) \in N \times N \mid n_1 \sim n_2\}$ is a closed analytic subvariety in $N \times N$. Assume that $N \xrightarrow{q} M$ is an unramified Galois covering with Galois group Π and $\sim$ is an analytic Π-invariant equivalence relation. We say that $\sim$ is a q-relation if $\Gamma \stackrel{\text{def}}{=} q(\tilde{\Gamma}) \subset M \times M$ is a closed subset. In this case Γ defines an analytic equivalence relation $\sim_M$ on M. We say that $\sim$ is a q-proper equivalence relation if for any compact $C \subset N$ the map $q \circ p_2 : (p_1^{-1}(C) \cap \tilde{\Gamma}) \to M$ is a proper map, where $p_1, p_2 : N \times N \to N$ are the natural projections.

LEMMA 5. *Any q-proper equivalence relation is a q-relation.*

PROOF. Clear. □

If $\sim$ is an analytic equivalence relation on N and $C \subset N$ is a compact analytic subvariety, we define $C_\sim = \{n \in N \mid \exists c \in C \text{ s.t. } c \sim n\}$. It is clear that $C_\sim$ is an analytic subvariety in N. If $C = \{n\}$ we will write Ω_n instead of $\{n\}_\sim$.

PROPOSITION 1. *Let $\tilde{\Lambda} \subset D^\sigma$ be a G^σ-invariant analytic subvariety $\tilde{\Lambda} \neq D^\sigma$ and $\sim$ be a G^σ-invariant analytic q-relation on $N \stackrel{\text{def}}{=} D^\sigma - \tilde{\Lambda}$ where $q : N \to M \stackrel{\text{def}}{=} X^\sigma - p^\sigma(\tilde{\Lambda})$ is the restriction of p^σ on N. Then there exists a G^σ-invariant analytic subvariety $\Lambda \subset N$ such that* $\dim \Lambda \leqq \dim Y$ *and* $\dim \Omega_n = 0$ *for any $n \in N - \Lambda$.*

PROOF. Define $N_S = \{n \in N \mid \Omega_n$ is singular at $n\}$. For any $n \in N - N_S$ define $C(n) = \dim T_{\Omega_n}(n)$ and take $C = \min_{n \in N - N_S} C(n)$. Define $N_C = \{n \in N - N_S \mid C(n) > C\}$ and take $N_0 = N_C \cup N_S$. It is clear that N_0 is an analytic G^σ-invariant subvariety of $\tilde{N} = D^\sigma - \tilde{\Lambda}$. As follows from Corollary 2 to Lemma 3, $\dim \tilde{\Lambda} < \frac{1}{2} \dim Y^\sigma$. The same arguments show that $\dim N_0 < \frac{1}{2} \dim Y^\sigma$. Consider the subbundle $\tilde{W} \subset T_{D_0^\sigma}$ on $D_0^\sigma \stackrel{\text{def}}{=} N - N_0$ given by $\tilde{W}\,|_{d^\sigma} = T_{\Omega_{d^\sigma}}(d^\sigma)$, where the vertical stroke stands for "restriction".

LEMMA 6. *There exists $J \subset [1, \cdots, n]$ such that $\tilde{W} = \bigoplus_{i \in J} T_{i,D^\sigma}\,|_{D_0^\sigma}$.*

PROOF OF LEMMA. Let α be the imbedding $\alpha : D_0^\sigma \to N$ and $\tilde{W}' \overset{\text{def}}{=} \alpha_*(\tilde{W})$ be the direct image. As codim $N_0 >$ codim $Y^\sigma > 2$ and $\tilde{W}$ is a locally free sheaf on D_0^σ, $\tilde{W}$ is a coherent G^σ-sheaf on N ([13]). Consider $\tilde{W}_1 \overset{\text{def}}{=} \tilde{W}^{\vee\vee}$ the double dual of $\tilde{W}'$. It is a reflexive ([4]) G^σ-sheaf. Let $j : N \to D^\sigma$ be the natural imbedding, $\tilde{W}_1' = j_*(\tilde{W}_1)$. As $\tilde{W}_1$ is reflexive and codim $\bar{\Lambda} > 2$, we see ([13]) that $\tilde{W}_1'$ is a coherent G^σ-sheaf on D^σ.

Consider $\tilde{W}_2 = \tilde{W}_1'^{\vee\vee}$ and the corresponding sheaf W_2 on X^σ. W_2 is a reflexive sheaf and therefore $V \overset{\text{def}}{=} j_*^\sigma(W_2)$ is a coherent analytic sheaf on $\bar{X}^\sigma$ (as before $j^\sigma : X^\sigma \to \bar{X}^\sigma$ is the natural imbedding). By [12] V corresponds to an algebraic sheaf V_a on $\bar{X}_a^\sigma$ and therefore W_2 corresponds to an algebraic sheaf $W_{2,a}$ on X^σ. Let $Z^\sigma = \{z^\sigma \in X^\sigma \mid W_2$ is not locally free at $z^\sigma\}$. Then Z^σ is an algebraic subvariety in X^σ and $\tilde{Z}^\sigma \overset{\text{def}}{=} p^{\sigma^{-1}}(Z^\sigma) \subset D^\sigma$ is G^σ-invariant. By Lemma 3, $\tilde{Z}^\sigma = \varnothing$ and therefore $\tilde{W}_2$ is locally free.

By the construction we have an imbedding $\tilde{W}_2\big|_{D_0^\sigma} \xrightarrow{\tilde{\varphi}_0} T_{D^\sigma}\big|_{D_0^\sigma}$. As codim$(D^\sigma - D_0^\sigma) > 1$ we may extend $\tilde{\varphi}_0$ to $\tilde{\varphi} : \tilde{W}_2 \to T_{D^\sigma}$. Consider $\varphi : W_2 \to T_{X^\sigma}$ and $j_*^\sigma(\varphi)$: $V \to j_*^\sigma(T_{D^\sigma})$. As we have seen before, V and (analogously) $j_*^\sigma(T_{D^\sigma})$ are algebraic sheafs. Let $Z_1 = \{z_1 \in X^\sigma \mid \varphi(W_2)$ is not a subbundle of T_{X^σ} in any neighbourhood of $z_1\}$. Then $Z_1 \subset X^\sigma$ is an algebraic subvariety and $\tilde{Z}_1 \overset{\text{def}}{=} p^{\sigma^{-1}}(Z_1)$ is G^σ-invariant. Therefore, by Lemma 3, $Z_1 = \varnothing$ and so $\tilde{W}_2$ is a G^σ-invariant subbundle of T_{D^σ}.

Let now $L_a \subset T_{X_a}$ be the algebraic subsheaf corresponding to $W_{2,a} \subset T_{X_a^\sigma}$. Consider $\tilde{L} = p^*(L) \subset T_D$. It is clear that $\tilde{L} \subset T_D$ is a $G_{\mathbf{Q}}$-invariant subbundle of D_X. By lemma 10 in [5] there exist $J \subset [1, \cdots, n]$ such that $\tilde{L} = \bigoplus_{i \in J} T_{i,D}$. It is clear now that $\tilde{W} = \tilde{W}_2\big|_{D_0^\sigma} = \bigoplus_{i \in J} T_{i,D^\sigma}$. Lemma 6 is proved. □

Let $H \subset G$ be a subgroup which satisfies the conditions of Proposition 2.1 and $'D_H^\sigma \subset D^\sigma$ be defined as in §1. Consider $'D_H^\sigma \cap \bar{\Lambda}$. As dim $\bar{\Lambda} \leqq$ dim $Y <$ dim $'D_H^\sigma$ it is a proper $'H^\sigma$ invariant subvariety in $'D_H^\sigma$. It follows now from Corollary to Lemma 1.2 that $'D_H^\sigma \cap \bar{\Lambda} = \varnothing$. The same arguments show that $'D_H^\sigma \subset D_0^\sigma$. It is clear now that the restriction $\sim_H$ of the equivalence relation $\sim$ on $'D_H^\sigma$ such that $\tilde{W}_H = \bigoplus_{i \in J} T_{i,'D_H^\sigma}$ where $\tilde{W}_H$ and $T_{i,'D_H^\sigma}$ are subbundles in $T_{'D_H^\sigma}$ defined analogously to $\tilde{W}$ and T_{i,D^σ}. Since the Main Theorem is known for H it follows from the proof of lemma 11 in [5] that $J = \varnothing$. Therefore $\Omega_n \subset N$ is a discrete set for $n \in D_0^\sigma$. Define $\Lambda = \{n \in N \mid \dim \Omega_n > 0\}$. Since $\Lambda \subsetneqq N$ is a G^σ-invariant analytic subvariety we have dim $\Lambda \leqq$ dim Y. Proposition 1 is proved.

§4. Let $\mathscr{H}$ be a complex Hilbert space. We denote by $\mathscr{A}(\mathscr{H})$ the graded algebra of polynomial functions on $\mathscr{H}$. Let $\mathbf{P}(\mathscr{H}) \overset{\text{def}}{=} \mathscr{H}^* - \{0\}/\mathbf{C}^*$ be the quotient space. Then $\mathbf{P}(\mathscr{H}) \subset \operatorname{Proj}(\mathscr{A}(\mathscr{H}))$ and this induces the structure of a ringed space on $\mathbf{P}(\mathscr{H})$.

On the other hand, $\mathscr{H}$ has a natural structure of a metric space. Let $S = \{h \in \mathscr{H} \mid \|h\| = 1\}$ and $C = \{z \in \mathbf{C} \mid |z| = 1\}$. Then C naturally acts on the metric space $\mathscr{H}$ and $\mathbf{P}(\mathscr{H}) = S/C$. So we can define a metric $d(\cdot,\cdot)$ on $\mathbf{P}(\mathscr{H})$ as a quotient metric from S. This metric d comes from the Riemannian metric ρ on $\mathscr{H}$ such that for any point $V \in \mathbf{P}(\mathscr{H})$ (i.e., for a line V in $\mathscr{H}$) the corresponding quadratic form ρ_V on $T_V(\mathbf{P}(\mathscr{H})) \simeq \mathscr{H}/V$ is given by $\rho_V(V', V'') \overset{\text{def}}{=} \langle V', V''\rangle$ where we identify $\mathscr{H}/V$ with the orthogonal complement of V in $\mathscr{H}$.

We denote by $\mathscr{H}^*$ the dual space to $\mathscr{H}$. Then points of $\mathbf{P}(\mathscr{H}^*)$ are subspaces of codimension one in $\mathscr{H}$.

Let X be a connected complex analytic manifold, $\Omega(=\Omega_X)$ be the sheaf of holomorphic differential forms of degree $n = \dim X$ and $\hat{\Omega}(=\hat{\Omega}_X)$ be the sheaf of germs of measurable sections of Ω. We denote by $\mathscr{H}(X) \subset H^0(X,\Omega)$ the subspace of forms $\omega \in H^0(X,\Omega)$ such that $\int_X \omega \wedge \bar{\omega} < \infty$ and define the scalar product $(\omega_1, \omega_2) \overset{\text{def}}{=} \int_X \omega_1 \wedge \omega_2$ for $\omega_1, \omega_2 \in \mathscr{H}(X)$. We also define $\hat{\mathscr{H}}(X) = \{\hat{\omega} \in H^0(X,\hat{\Omega}) \mid \int_X \hat{\omega} \wedge \bar{\hat{\omega}} < \infty\}$. It is well known (see [6]) that $\mathscr{H}(X)$ is a closed subspace of a Hilbert space $\hat{\mathscr{H}}(X)$. We denote by $\mathbf{P}(X)$ the projective space $\mathbf{P}(\mathscr{H}(X)^*)$.

Consider now X as a $2r$-dimensional C^∞-manifold and define the section μ_X of $\Omega \wedge \bar{\Omega}$ over X by

$$\mu_X(x) = \sup_{\substack{\omega \in \Omega(X) \\ (\omega,\omega)=1}} (\omega \wedge \bar{\omega})_x.$$

It is called the Bergman volume on X. It is clear that μ_X is a nonnegative smooth section of $\Omega \wedge \bar{\Omega}$ which is invariant under the group $G(X)$ of analytic automorphisms of X. We will consider μ_X as a measure on X.

LEMMA 1. $\int_X \mu_X = \dim \mathscr{H}(X)$.

PROOF. Let $\omega_1, \cdots, \omega_N$, $1 \leqq N \leqq \infty$, be an orthonormal basis for $\mathscr{H}(X)$. Then

$$\mu_X = \sum_{i=1}^{N} \omega_i \wedge \bar{\omega}_i. \qquad \square$$

LEMMA 2. *If $X' \subset X$ is an open subset, then $\mu_{X'} \geqq \mu_X$ on X'.*

PROOF. Follows immediately from the definition. $\square$

Assume now that $\mathscr{H}(X) \neq \{0\}$.

For any $\omega \in \mathscr{H}(X)$ we define X_ω = Divisor of zeros of ω. Consider $V \overset{\text{def}}{=} \bigcap_{\omega \in \mathscr{H}(X)} X_\omega$. It is clear that $V = \{\text{Zeros of } \mu_X\}$. Let $X_0 = X - V$. For any point $x \in X_0$ we denote by $\mathscr{H}_x \subset \mathscr{H}(X)$ the subspace of $\omega \in \Omega(X)$ such that

$\omega_x = 0$. It is clear that $\mathcal{H}_x$ is a subspace of codimension one in $\mathcal{H}(X)$. That is, we have a canonical map

$$\varphi_0 : X_0 \to \mathbf{P}(X).$$

It is clear that φ_0 is analytic (i.e., φ_0 is a map of ringed spaces). Consider $\rho_x \stackrel{\text{def}}{=} \varphi_0^*(\rho)$. ρ_X is by the definition a pseudometric on X_0 which is called the Bergman pseudo-metric. It is clear that

$$\rho_X = \frac{\partial^2}{\partial z \partial \bar{z}} \ln \mu_X.$$

LEMMA 3. *Let X be an open subspace of a complete complex manifold $\bar{X}$ such that $Y \stackrel{\text{def}}{=} \bar{X} - X$ is a divisor with normal crossing. Then the restriction map $\alpha : \mathcal{H}(\bar{X})$ is an isomorphism.*

PROOF. It is clear that α is an imbedding. Now take some $\omega \in \mathcal{H}(X)$. Let $Y_0 \subset Y$ be the set of smooth points of Y and $y \in Y_0$. Choose a neighbourhood U of y in $\bar{X}$ such that $(U, U_0) \simeq (D^r, D^{r-1} \times D^*)$ where $U_0 = U \cap X$, $D = \{z \in \mathbf{C} \mid |z| \leqq 1\}$, $D^* = D - \{0\}$. Then

$$\omega \big|_{U_0} = \varphi(z, t), dz_1 \wedge \cdots \wedge dz_{r-1} \wedge dt$$

where $z \in D^{r-1}, t \in D^*$ and

$$\int_{U_0} \omega \wedge \bar{\omega} = \int_{D^{r-1} \times D^*} |\varphi(z,t)|^2 d\mu < \infty$$

where $d\mu$ is the Lebesgue measure on $D^{r-1} \times D^*$. We can write $\varphi(z,t) = \sum_{l=-\infty}^{\infty} t^l \varphi_l(z)$. It is clear now that the condition $\int_{D^{n-1} \times D^*} |\varphi(z,t)|^2 d\mu < \infty$ implies $\varphi_l(z) \equiv 0$ for $l < 0$. Therefore ω is regular on Y_0. As $\operatorname{codim}_{\bar{X}}(Y - y_0) > 1$ we see that ω is regular everywhere. □

COROLLARY. *Let X_a be an algebraic k-manifold and σ_1, σ_2 be two imbeddings of k to $\mathbf{C}$. Then* $\dim \mathcal{H}(X_1) = \dim \mathcal{H}(X_2)$ *where* $X_i = (X_a \bigotimes_{\sigma_i} \operatorname{Spec} \mathbf{C})_{\mathbf{C}}$.

Assume now that X is an open dense subset of a complete analytic space $\bar{X}$ and $Y \stackrel{\text{def}}{=} \bar{X} - X$ is an analytic subvariety.

PROPOSITION 1. *For any $\varepsilon > 0$ there exists a compact $C \subset X$ such that for any unramified covering $\varphi : Z \to X$ we have $\int_{\varphi^{-1}(U)} \mu Z < \varepsilon \cdot \deg \varphi$, where $U = X - C$.*

Let $D(a) = \{z \in \mathbf{C} \mid |z| < a\}$, $D^*(a) = D(a) - \{0\}$. The proof of Proposition 1 is based on the following elementary

LEMMA 4. *Consider the map $\varphi_n : D^*(1) \to D^*(1)$ given by $z \to \bar{z}^n$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_\varphi \int_{\varphi_n^{-1}(D^*(\delta))} \mu_{D^*(1)}/n < \varepsilon \qquad \text{for all } n.$$

PROOF. As is well known,

$$\mu_{D^*(1)} = \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2}.$$

Therefore

$$\int_{\varphi_n^{-1}(D^*(\delta))} \mu_{D^*(1)} = 2\pi \int_0^{\delta^{1/n}} \frac{tdt}{(1-t^2)^2} < \frac{20n}{\ln 1/\delta}. \qquad \square$$

COROLLARY 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any (possibly disconnected) unramified covering $\varphi : Z \to D^*(1)$ we have $\int_{\varphi^{-1}(D^*(\delta))} \mu_Z < \varepsilon \cdot \deg \varphi$.*

COROLLARY 2. *Let $\bar{X} = D(1)^r$, $X = D^*(1)^r$, $Y = \bar{X} - X$. For any $\varepsilon > 0$ there exists an open neighbourhood U of Y in $\bar{X}$ such that for any unramified covering $\varphi : Z \to X$ we have $\int_{\varphi^{-1}(U)} \mu_Z < \varepsilon \cdot \deg \varphi$.*

Now we can prove Proposition 1. Because the statement of the proposition involves only X we can assume that $\bar{X}$ is nonsingular and Y is a divisor with normal crossing in $\bar{X}$. Therefore, we can find a finite number of open sets $V_i \subset \bar{X}$, $1 \leqq i \leqq N$, such that:

(a) For any i, $1 \leqq i \leqq N$, the pair $(V_i, V_i \cap X)$ is isomorphic to $(D(1)^r, D(1)^k \times D^*(1)^{r-k})$ for some k, $0 \leqq k < r$.

(b) $\bigcup_{i=1}^N V_i \supset Y$.

Let $U_i \subset V_i$, $1 \leqq i \leqq N$ be open sets as in Corollary 2. Take $U = \bigcup_{i=1}^N U_i$. Then U is an open neighbourhood of Y in $\bar{X}$ and for any unramified covering $\varphi : Z \to X$ we have

$$\int_{\varphi^{-1}(U)} \mu_Z \leqq \sum_{i=1}^n \int_{\varphi^{-1}(U_i)} \mu_Z \leqq \sum_{i=1}^N \int_{\varphi^{-1}(U_i)} \mu_{\varphi^{-1}(V_i)} \leqq (\varepsilon N) \deg \varphi.$$

It is clear that we can take $C = X - U$. Proposition 1 is proved. $\square$

COROLLARY (to the proof). *Let $\varphi : Z \to X$ be any unramified Galois covering. Then $\int_X d\mu_Z < \infty$, where $d\mu_Z$ is considered as a volume form on X.*

Now let X be as in Proposition 1 and let $p : \tilde{X} \to X$ be an infinite unramified

Galois covering with Galois group Γ. Let $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_l \supset \cdots$ be a sequence of normal subgroups of finite index in Γ such that $\bigcap \Gamma = (e)$.

We define $X_l = \tilde{X}/\Gamma_l$ and $h_l = \dim \mathcal{H}(X_l)$.

THEOREM 1. (a) *$h_l < \infty$ for any l.*

(b) *The sequence $h_l/[\Gamma:\Gamma_l]$ is bounded.*

(c) *If $h_l/[\Gamma:\Gamma_l] \not\to 0$ then $\mathcal{H}(\tilde{X}) \neq \{0\}$.*

PROOF. For any l, μ_{X_l} is a Γ-invariant measure on X_l. Therefore $\mu_{X_l} = q_l^*(\mu_l)$ where $q_l : X_l \to X$ is the canonical projection and μ_l is a measure on X. It follows from Lemma 1 that $\int_X \mu_l = h_l/[\Gamma:\Gamma_l]$.

To prove (a) and (b) take $\varepsilon = 1$ and choose C as in Proposition 1.

As C is compact, we can find a finite number of subsets $U_i \subset X$, $1 \leqq i \leqq N$ such that

(a) there exists an analytic isomorphism $\varphi_i : U_i \xrightarrow{\sim} D(1)^r$,

(b) $\bigcup_{i=1}^N \varphi_i^{-1}(D(\frac{1}{2})^r) \supset C$.

It follows now from Lemma 2 that $\mu_{l|_C} \leqq \nu_{|_C}$ where ν is a measure on $\bigcup_{i=1}^N \varphi_i^{-1}(D(\frac{1}{2})^r)$ given by $\nu = \sum_{i=1}^N \varphi_i^*(\mu_{D(1)^r})$.

It is clear that $\int_C \nu \leqq N \int_{D(1/2)^2} \mu_{D(1)^r} < \infty$.

Therefore

$$h_l/[\Gamma:\Gamma_l] = \int_X \mu_l = \int_C \mu_l + \int_{X-C} \mu_l \leqq \int_C \nu + 1.$$

This proves (a) and (b).

Assume now that $h_l/[\Gamma:\Gamma_l] \not\to 0$. Choosing a subsequence we may assume that $h_l/[\Gamma:\Gamma_l] \geqq a > 0$ for all l. Take $\varepsilon = a/2$ and choose a compact C as in Proposition 1. Then

$$\int_C \mu_l = \int_X \mu_l - \int_{X-C} \mu_l = h_l/[\Gamma:\Gamma_l] - \int_{X-C} \mu_l \geqq \varepsilon.$$

Consider now the sequence of functions μ_l/ν on C. As C is compact, $\mu_l \leqq \nu$ and $\varepsilon \leqq \int_C \mu_l$ we may assume (choosing a subsequence) that there exists a sequence of points $X_l \in C$ such that

$$x_l \to x_0 \quad \text{and} \quad \frac{\mu_l}{\nu}(x_l) \to a > 0.$$

It is clear then that

$$\frac{\mu_l}{\nu}(x_0) \to a \quad \text{for} \quad l \to \infty.$$

Fix now a point $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Consider a complete Riemannian metric ρ on X and denote by $\tilde{\rho}$ the induced metric on $\tilde{X}$. Define

$$\tilde{X}_l = \{\tilde{x} \in X \mid \tilde{\rho}(\tilde{x}_0, \tilde{x}) < \tilde{\rho}(\tilde{x}_0, \gamma\tilde{x}) \,\forall \gamma \in \Gamma_l - \{e\}\}.$$

$\tilde{X}_l$ is a fundamental domain for the action of Γ_l on $\tilde{X}$. Then the projection $p_l : \tilde{X} \to \tilde{X}_l$ induces an isomorphism between $\tilde{X}_l$ and an open dense subset in X_l.

Now we can prove Theorem 1 (c). By the definition for any l there exists $\omega_l \in \mathcal{H}(X_l)$ such that $(\omega_l, \omega_l) = 1$ and $(\omega_l \wedge \bar{\omega}_l)(p_l(\tilde{x}_0)) = q_l^*(\mu_l)$. Therefore (see §3 in [5]) there exists a compact neighbourhood $\tilde{U}$ of $\tilde{x}_0$ in $\tilde{X}$ and $b > 0$ such that $\int_{p_l(U)} \omega_l \wedge \bar{\omega}_l \geqq b$.

Consider now the Hilbert space $\hat{\mathcal{H}}$ of measurable square-integrable sections η of $\Omega_{\tilde{X}}$ and define $\eta_l \in \hat{\mathcal{H}}$ by $\eta_l \mid \tilde{X}_l = p_l^*(\omega_l)$, $\eta_l \mid \tilde{X} - \tilde{X}_l = 0$.

Since $(\eta, \eta) = 1$ we can find a weakly convergent subsequence $\eta_{l_i} \to \eta$, $\eta \in \hat{\mathcal{H}}$. It is easy to see that η is holomorphic on $\bigcup_l \tilde{X}_l = \tilde{X}$ (i.e., $\eta \in \mathcal{H}(\tilde{X})$) and that $\int_{\tilde{X}} \eta \wedge \bar{\eta} \geqq b$. So $\mathcal{H}(\tilde{X}) \neq \{0\}$. □

Consider now $\tilde{V} \overset{\text{def}}{=} \bigcup_{\omega \in \mathcal{H}(\tilde{X})} \tilde{X}_\omega$, $N = \tilde{X} - \tilde{V}$. It is clear that $\tilde{V}$ is a Γ-invariant analytic subvariety in $\tilde{X}$ and therefore $V \overset{\text{def}}{=} \Gamma(\tilde{V})$ is an analytic subvariety in X. Let $M = X - V$ and $q : N \to M$ be the restriction of p on N. We define now a Γ-invariant equivalence relation $\sim$ on N by saying that

$$n_1 \sim n_2 \Leftrightarrow \text{for any } \omega_1, \omega_2 \in \mathcal{H}(\tilde{X}) \text{ such that } n_1, n_2 \notin X_{\omega_2} \text{ we have}$$

$$\frac{\omega_1}{\omega_2}(n_1) = \frac{\omega_1}{\omega_2}(n_2).$$

It is clear that $\sim$ is an analytic Γ-invariant relation on N.

PROPOSITION 2. *$\sim$ is a q-proper equivalence relation.*

To clarify the ideas of the proof we consider first the special case. For any $n \in N$ consider $\Omega_n = \{n' \in N \mid n \sim n'\}$.

LEMMA 5. *Let $n \in N$ and $\Gamma_n = \{\gamma \in \Gamma \mid \gamma n \in \Omega_n\}$. Then Γ_n is finite.*

PROOF OF LEMMA 5. As $\sim$ is a Γ-invariant relation we have $(\Omega_n) \cdot \gamma = \Omega_{n\gamma}$ for any $\gamma \in \Gamma$. Therefore $\Gamma_n = \mathrm{St}_\Gamma(\Omega_n)$ and it is a subgroup in Γ. Fix now some $\omega \in \mathcal{H}(\tilde{X})$ such that $n \notin \tilde{X}_\omega$ and define a function $\chi : \Gamma \to \mathbf{C}$ by

$$\chi(\gamma) = \left.\frac{\omega^\gamma}{\omega}\right|_n, \quad \gamma \in \Gamma.$$

For $\gamma_1 \in \Gamma_n$, $\gamma_2 \in \Gamma$ we have

$$\chi(\gamma_1\gamma_2)=\frac{\omega^{\gamma_1\gamma_2}}{\omega}\Big|_n=\left(\frac{\omega^{\gamma_1\gamma_2}}{\omega^{\gamma_1}}\Big|_n\cdot\frac{\omega^{\gamma_1}}{\omega}\Big|_n\right)$$
$$=\frac{\omega^{\gamma_2}}{\omega}\Big|_{n\gamma_1}\circ\frac{\omega^{\gamma_1}}{\omega}\Big|_n=\chi(\gamma_2)\cdot\chi(\gamma_1)$$

because $n\gamma_1\sim n$. Therefore the restriction of χ on Γ_n is the multiplicative character $\chi:\Gamma_n\to\mathbf{C}^*$.

It is clear that $\chi:\Gamma_n\to\mathbf{C}^*$ does not depend on the choice of $\omega\in\mathcal{H}(\tilde{X})$.

Fix an open neighbourhood U of n in $\tilde{X}$ such that $(U,n)\xrightarrow[\varphi]{\sim}(D(1)r,0)$, and $U\gamma\cap U=\emptyset$ for any $\gamma\in\Gamma-\{e\}$. Fix also $\omega_0\in\mathcal{H}(\tilde{X})$ such that $n\notin\tilde{X}_{\omega_0}$. Then for any $\omega\in\mathcal{H}(X)$ we have

$$(\omega,\omega)\geqq\sum_{\gamma\in\Gamma}\int_U\omega^\gamma\wedge\bar{\omega}^\gamma$$

and there exists $c>0$ such that

$$\int_U\omega\wedge\bar{\omega}\geqq c\left|\frac{\omega}{\omega_0}(n)\right|\qquad\text{for any }\omega\in\mathcal{H}(\tilde{X}).$$

Therefore we have

$$\sum_{\gamma\in\Gamma_n}|\chi(\gamma)|^2\leqq\frac{1}{c}\sum_{\gamma\in\Gamma_n}\int_U\omega_0^\gamma\wedge\bar{\omega}_0^\gamma\leqq\frac{1}{c}(\omega_0,\omega_0)<\infty$$

and so Γ_n is finite. □

Consider now the general case. Let $C\subset N$ be a compact

$$\Lambda_C=\{n\in N\mid\text{s.t.}\ \exists c\in C\ \text{s.t.}\ n\sim c\}.$$

We have to prove that $a_C:\Lambda_C\to M$ is a proper map.

Assume that q is not proper. Then we can find two sequences (n_i) and $(n_i')\in N$, $1\leqq i<\infty$ and two points $n\in N$ and $m\in M$ such that

(a) $\{(n_i)\}\subset N$ is a discrete subset,

(b) $n_i'\to n_0'\in N$ for $i\to\infty$,

(c) $n_i\sim n_i'$ for all $i\geqq1$,

(d) $q(n_i)\to m$ for $i\to\infty$.

Choose now an open neighbourhood U of m in M such that (U,m) is analytically isomorphic to $(D(1)^r,0)$.

We may assume that $q(n_i)\in U$ for all i.

Let $\tilde{U}\subset N$ be a connected component of $q^{-1}(U)$ such that $\Omega_{n_0}\cap\tilde{U}\neq\emptyset$ and fix $n\in\Lambda_{n_0}\cap\tilde{U}$. Define $y_0=\tilde{U}\cap q^{-1}(m)$, $y_i=\tilde{U}\cap q^{-1}(q(n_i))$ for $i\geqq1$. Then for any $i\geqq1$ there exists $\gamma_i\in\Gamma$ such that $y_i=n_i\cdot\gamma_i$. We may assume that $\gamma_i\neq\gamma_j$ for

$i \neq j$. Fix now a nonvanishing C^∞-volume form ν on U and define $\tilde{\nu} = q^*(\nu)$ on $q^{-1}(U)$.

Also fix a nonvanishing C^∞-volume form $\tilde{\nu}'$ on a neighbourhood $\tilde{U}'$ of n'_n in N. By the definition of N we can find $\omega_0 \in \mathscr{H}(X)$ with $(\omega_0, \omega_0) = 1$ such that $n_0, n'_0, y \notin \tilde{X}_\omega$. As $y_i \to y_0$, and $n'_i \to n'_0$ for $i \to \infty$ we may assume that $n'_i \in U'$ for any $i \geqq 1$ and:

(a) There exists $\alpha > 0$ such that

$$\left(\frac{\omega_0 \wedge \bar{\omega}_0}{\nu}\right)(y_i) > \alpha \quad \text{and} \quad \left(\frac{\omega_0 \wedge \bar{\omega}_0}{\nu'}\right)(n'_i) > \alpha \quad \text{for any} \quad i \geqq 1.$$

(b) There exists $\beta > 0$ such that for any $i \geqq 1$ and any $\omega \in \mathscr{H}(\tilde{X})$

$$\int_{\tilde{U}} \omega \wedge \bar{\omega} \geqq \beta \frac{\omega \wedge \bar{\omega}}{\nu}(y_i) \quad \text{and} \quad \int_{\tilde{U}'} \omega \wedge \bar{\omega} \geqq \beta \frac{\omega \wedge \bar{\omega}}{\nu'}(n'_i).$$

Consider now the sequence

$$a_i \overset{\text{def}}{=} \frac{\omega_0 \wedge \bar{\omega}_0}{\tilde{\nu}}(n_i).$$

Then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\tilde{\nu}}(y_i) \leqq \frac{1}{\beta} \sum_{i=1}^{\infty} \int_U \omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}$$

$$= \frac{1}{\beta} \sum_{i=1}^{\infty} \int_{U^{\gamma^{-1}}} \omega_0 \wedge \bar{\omega}_0 \leqq \frac{1}{\beta} \int_{\tilde{X}} \omega_0 \wedge \bar{\omega}_0 = \frac{1}{\beta}.$$

Therefore $a_i \to 0$ for $i \to \infty$. On the other hand

$$\frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\omega_0 \wedge \bar{\omega}_0}(n_i) = \frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\omega_0 \wedge \bar{\omega}_0}(n'_i)$$

and therefore

$$\frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\tilde{\nu}'}(n'_i) = \frac{\omega_0 \wedge \bar{\omega}_0}{\tilde{\nu}'}(n'_i) \cdot \frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\tilde{\nu}}(n_i) \left(\frac{\omega_0 \wedge \bar{\omega}_0}{\tilde{\nu}}(n_i)\right)^{-1}$$

$$= \frac{\omega_0 \wedge \bar{\omega}_0}{\tilde{\nu}'}(n'_i) \cdot \frac{\omega_0 \wedge \bar{\omega}_0}{\nu}(y_i) \cdot \left(\frac{\omega_0 \wedge \bar{\omega}_0}{\tilde{\nu}}(n_i)\right)^{-1} \geqq \frac{\alpha^2}{a_i}.$$

Therefore

$$\int_{\tilde{X}} \omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i} \geqq \int_{\tilde{U}} \omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i} \geqq \beta \frac{\omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i}}{\tilde{\nu}'}(n'_i)$$

$$\geqq \frac{\alpha^2 \beta}{a_i} \to \infty \qquad \text{for} \quad i \to \infty.$$

But

$$\int_{\tilde{X}} \omega_0^{\gamma_i} \wedge \bar{\omega}_0^{\gamma_i} = \int_{\tilde{X}} \omega_0 \wedge \bar{\omega}_0 = 1.$$

This contradiction proves Proposition 2.

§5. In this paragraph we will finish the proof of the Main Theorem.

Let G be a nonclassical simple $\mathbf{Q}$-group such that $D = K \setminus G_{\mathbf{R}}$ is a Hermitian symmetric space. Let $\Gamma \subset G_{\mathbf{Q}}$ be an arithmetic subgroup without elements of finite order. Define X, X^σ, Γ^σ, G^σ and D^σ as in §1.

THEOREM 1. $\mathcal{H}(D^\sigma) \neq \{0\}$.

PROOF. Choose a sequence $\Gamma = \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots$ of normal subgroups of finite index in Γ such that $\bigcap \Gamma_n = (e)$. Then we can define the sequence $\Gamma^\sigma = \Gamma_1^\sigma \supset \cdots \Gamma_n^\sigma \supset \cdots$ of normal subgroups in Γ^σ such that $X_n^\sigma = D^\sigma / \Gamma_n^\sigma$ where $X_n = D/\Gamma_n$. Define $h_n = \dim H(X_n)$, $h_n^\sigma = \dim H(X_n^\sigma)$.

LEMMA 1. *$h_n^\sigma / [\Gamma^\sigma : \Gamma_n^\sigma] \not\to 0$ for $n \to \infty$.*

PROOF OF LEMMA 1. It is clear that $[\Gamma^\sigma : \Gamma_n^\sigma] = [\Gamma : \Gamma_n]$ and as follows from the corollary to Lemma 4.3, $h_n^\sigma = h_n$. So we have to prove that $h_n / [\Gamma : \Gamma_n] \not\to 0$, for $n \to \infty$.

We will use the following result from the theory of representations which we will prove in another paper.

THEOREM A. *Let $G_{\mathbf{R}}$ be a semisimple real group, $\Gamma \subset G_{\mathbf{R}}$ be an arithmetic subgroup, $\Gamma = \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots$ be a sequence of normal subgroups of finite index in Γ such that $\bigcap \Gamma_n = \{e\}$. Let (σ, W) be an irreducible cuspidal representation of $G_{\mathbf{R}}$. Define $h_n(\sigma) \stackrel{\text{def}}{=} \dim \operatorname{Hom}_{G_{\mathbf{R}}}(W, L^2(G_{\mathbf{R}}/\Gamma_n))$. Then*

$$\frac{h_n(\sigma)}{[\Gamma : \Gamma_n]} \not\to 0.$$

Consider now $W \stackrel{\text{def}}{=} \mathcal{H}(D)$ and let σ be the natural action of $G_{\mathbf{R}}$ on W. We can consider σ as a unitary representation of $G_{\mathbf{R}}$ and it is well known that σ is cuspidal.

LEMMA 2. $h_n = h_n(\sigma)$.

PROOF. We can realize D as a bounded open subset in $\mathbf{C}^n$ in such a way that K acts linearly on $\mathbf{C}^n \cdot k \to \gamma(k)$.

Take $\omega_0 = dz_1 \wedge \cdots \wedge dz_n$. Then $\omega_0 \in \mathcal{H}(D)$, $\tau(k)\omega_0 = \chi(k)\omega_0$, where $\chi(k) = \det \gamma(k)$ and ω_0 is determined up to a scalar by this property. Define

$$L^{\chi}(G/\Gamma_n) = \{f \in L^2(G/\Gamma_n) \mid f(k\dot{g}) = \chi(k)f(\dot{g})\}, \quad \forall k \in K, \quad \dot{g} \in g/\Gamma_n.$$

As is well known, $L^{\chi}(G/\Gamma_n) \xrightarrow[\varphi]{\sim} \{L^2\text{-sections of } \Omega \text{ over } D/\Gamma_n\}$. Consider now the map $\beta : \mathrm{Hom}_{G_{\mathbf{R}}}(W, L^2(G/\Gamma_n)) \to \{L^2\text{-sections of } \Omega \text{ over } D/\Gamma_n\}$,

$$\beta(h) = \varphi(h(\omega_0)) \qquad \text{for } h \in \mathrm{Hom}_{G_{\mathbf{R}}}(W, L^2(G/\Gamma_n)).$$

It is clear that $\beta(h) \subset \mathcal{H}(X_n)$ and it is not difficult to prove that

$$\beta : \mathrm{Hom}_{G_{\mathbf{R}}}(W, L^2(X/\Gamma_n)) \to \mathcal{H}(X_n)$$

is an isomorphism. □

Now Lemma 1 follows immediately from Theorem A and Lemma 2 and Theorem 1 follows from Theorem 4.1 and Lemma 1. □

We will assume from now on that the Main Theorem is known for all groups H with $\dim H < \dim G$.

Consider the Bergman volume μ_{D^σ} on D^σ and define $\tilde{V} = \{\text{zeros of } \mu_{D^\sigma}\}$ as in §4. Then $\tilde{V} \subsetneq D^\sigma$ is a G^σ-invariant analytic subvariety.

If G is an anisotropic group, then it follows from Lemma 3.3 that $\tilde{V} = \emptyset$. If G is isotropic, we choose a pair (H, φ) which satisfies conditions of Proposition 2.1 and consider a subvariety $'D^\sigma_H \subset D^\sigma$ defined in Lemma 1.2.

LEMMA 3. $'D^\sigma_H \cap \tilde{V} = \emptyset$.

PROOF. It is clear that $'D^\sigma_H \cap \tilde{V} \subset {'D^\sigma_H}$ is an H^σ-invariant subvariety. As we have assumed that the Main Theorem is true for H, the Corollary to Lemma 1.2 is applicable and we see that either $'D^\sigma_H \cap \tilde{V} = \emptyset$ or $'D^\sigma_H \subset \tilde{V}$. But the second possibility will contradict Corollary 1 of Lemma 3.3. □

Let $N = D^\sigma - \tilde{V}$, $M = X^\sigma - p^\sigma(\tilde{V})$, $q : N \to M$ be the restriction of p^σ and $\sim$ be the equivalence relation defined in §4. By Proposition 4.2 $\sim$ is a q-proper analytic equivalence relation and therefore (by Proposition 3.1) there exists a G-invariant analytic subset $\Lambda \subset N$ such that $\dim \Lambda \leqq \dim Y$ and $\Omega_n \subset N$ is a discrete subset for $n \in N_0 \overset{\text{def}}{=} N - \Lambda$. By the definition the restriction of the Bergman pseudometric ρ_{D^σ} on N_0 is a nondegenerate metric.

LEMMA 4. $'D^\sigma_H \cap \Lambda = \emptyset$.

PROOF. As in the proof of Lemma 3 we see that $'D_H \cap \Lambda \subset {'D^\sigma_H}$ is either empty or is $'D^\sigma_H$ itself. But the second possibility is excluded because $\dim ({'D^\sigma_H} \cap \Lambda) \leqq \dim \Lambda \leqq \dim T < \dim {'D^\sigma_H}$. □

Fix now a CM-point $d_o \subset 'D_H \subset D$ and let $D_0^\sigma \in 'D_H^\sigma \subset D^\sigma$ be a point such that $p^\sigma(d_0^\sigma) = (p(d_0))^\sigma$ and consider the closure B of the orbit $G^\sigma D_0^\sigma$ in N_0. ρ_{D^σ} defines a structure of a metric space (N_0, ρ) of N_0. So we may consider (B, ρ) as a metric space.

Lemma 5. *There exists $\varepsilon > 0$ such that the ball $N_\varepsilon(d_0^\sigma)$ of radius ε around d_0^σ in N_0 is compact.*

Proof. Clear.

Proposition 1. *(B, ρ) is a complete metric space.*

Proof. By Lemma 5 we can find $\varepsilon > 0$ such that the ball $B_\varepsilon(d_0^\sigma)$ of radius ε around d_0^σ in B is compact. As ρ is G^σ-invariant a ball $B_\varepsilon(d) \subset B$ is compact for any $d \in \{G^\sigma d_0^\sigma\}$. Therefore $B_{\varepsilon/2}(b)$ is a compact ball for any $b \in B$. Proposition 1 is proved.

Corollary 1. *The group* Aut B *of isometries of (B, ρ) is a Lie group.*

Proof. It is clear that (B, ρ) is a finite-dimensional metric space. Therefore the corollary follows from Proposition 1 and the first corollary in §6.3 of [9]. □

Let G_B be the closure of G^σ in Aut B. As G_B is a closed subgroup in a Lie group, it is also a Lie group.

Lemma 6. *$B \subset N_0$ is a real submanifold.*

Proof. By the definition $G^\sigma(\Lambda_0^\sigma)$ is dense in B. Therefore ([9]) $G_B(d_0^\sigma) = B$. Lemma 6 now follows easily from Lemma 5.

Proposition 2. $B = D^\sigma$.

Proof. It follows from Lemma 3.1 that the tangent subspace $T_B(d_0^\sigma) \subset T_{D^\sigma}(d_0^\sigma)$ is invariant under multiplication by $\sqrt{-1}$. As $G^\sigma \subset G(D^\sigma)$ the same is true for any point $d \in \{G^\sigma d_0^\sigma\}$. Therefore for any point $b \in B$, $T_B(b)$ is a complex subspace in $T_{D^\sigma}(b)$ and consequently B is a closed G^σ-invariant analytic subvariety in N. By the construction $B \supset 'D_H^\sigma$, and therefore $\dim_\sigma B > \dim Y$. Using arguments from §3 we can easily conclude that $B = D^\sigma$. □

Now we can finish the proof of the Main Theorem. It follows from Proposition 2 that the conditions (a) and (b) from Proposition 1.1 are satisfied and it follows from the corollary to Lemma 4.4 that the condition (c) is also satisfied. □

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DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
CAMBRIDGE, MA 02138 USA